Challenges in assessing mathematical reasoning

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Mathematical reasoning is foundational to making sense of mathematics. Yet assessing mathematical reasoning can be challenging for teachers. This paper reports on a project where teachers taught two lessons with a specific focus on reasoning and came together with other teachers at their school to attempt to assess the reasoning of their students. Results, derived from an analysis of two post-lesson discussions about student work samples and associated completed assessment rubrics, provide insights into the challenges identified by 34 primary teachers at 4 Victorian government schools.

The inclusion of reasoning in the Australian Curriculum (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2017) has prompted the implementation of professional learning for primary teachers. A range of programs have been conducted, such as demonstration lessons (Herbert, Vale, Bragg, Loong, & Widjaja, 2015; Livy & Downton, 2018); support for peer-learning-teams (Herbert & Bragg, 2017); workshops (Hilton, Hilton, Dole & Goos, 2016); and development of resources for teachers to assess mathematical reasoning (Australian Academy of Science [AAS] and Australian Association of Mathematics Teachers [AAMT], 2018). As part of the reSolve project, a rubric for assessing reasoning was developed, the Assessing Mathematical Reasoning Rubric (AAS & AAMT, 2018) to assist teachers to notice and assess students' reasoning. The aim of this paper is to identify the challenges the 34 teachers faced in attempting to assess mathematical reasoning.

Literature review

Lannin, Ellis and Elliot (2011) described reasoning as "an evolving process of conjecturing, generalizing, investigating why, and developing and evaluating arguments", p. 13). Communication of ideas and discussion with other students or the teacher may assist in refining conjectures and convince them of the validity of conclusions (Brodie, 2010; Kilpatrick et al., 2001; Vale, Widjaja, Herbert, Bragg, & Loong, 2017). Brodie (2010) acknowledged the importance of reasoning "to understand mathematical concepts, to use mathematical ideas and procedures flexibly, and to reconstruct once understood, but forgotten mathematical knowledge" (p. 11). This occurs in classrooms where teachers and researchers have developed approaches intended to build mathematical understandings through problem solving activity (Wood et al., 2006) convincing them of the validity of their conclusions (Kilpatrick et al., 2001; Vale et al., 2017). At primary school this might involve choosing tasks which provide opportunities for a teacher to prompt students to think more deeply, expecting students to explain and justify their solutions and to look for patterns since this encourages students to form conjectures and verify their conjectures and solutions (Vale et al., 2017). However, in problem solving actions, such as the trial and error activities a student may use before a systematic search for examples, may not be noticed by the teacher (Ferrando, 2006). Students build new knowledge by creating and validating mathematical ideas through reasoning, thus building an appreciation of the connections between logical and meaningful mathematical notions as opposed to rote learning of disconnected routine procedures (Mata-Pereira, & da Ponte, 2017).

Considerable research has been conducted focussing on mathematical reasoning. However, the way mathematical reasoning is described "tends to be vague, unsystematic,

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and even contradictory from one document to the other" (Jeannotte & Kieran, 2017, p. 2). In addition, the diversity of research literature regarding reasoning amplifies the need for a holistic definition of mathematical reasoning (MR). They stated that "the research literature on MR suggest that this area is one that could benefit greatly from an attempt at coherent conceptualization" (p. 2). Their model consists of two parts: a structural aspect and a process aspect. The structural aspect is consistent with formal mathematical definitions of reasoning: deduction; induction; and abduction. These formal elements of reasoning are more applicable to senior secondary and tertiary mathematics. Jeannotte and Kieran's (2017) process aspect is more consistent with reasoning in primary schools. They identified nine distinct processes in the literature on mathematical reasoning.

> Eight of these were classified into one of two categories: the processes related to the search for similarities and differences, or the processes related to validating. … Five processes relate[d] to the search for similarities and differences: generalizing, conjecturing, identifying a pattern, comparing, and classifying. ... Three processes related to validation that emerged from the analysis of the corpus: justifying, proving, and formal proving. ... The ninth process that of exemplifying, was classified as a support for both of the other two categories (p. 9).

Kilpatrick et al.'s (2001) adaptive reasoning, that is, the "capacity for logical thought, reflection, explanation, and justification" (Kilpatrick et al., 2001, p. 5) is consistent with Jeannotte and Kieran's (2017) process aspect.

Some challenges to assessing mathematical reasoning have already been identified in the research literature: Teacher knowledge (Clarke, Clarke & Sullivan, 2012; Herbert et al., 2015; Loong et al., 2017) and noticing of reasoning (Llinares, 2013; Jacobs, Lamb, & Philipp, 2010; Francisco, & Maher, 2011); students' difficulties in articulating their reasoning (Bragg, et al., 2016); and planning for reasoning (Davidson, Herbert, & Bragg, 2018). Teachers' knowledge of the content they teach is crucial in effective teaching (Darling-Hammond, 2000) so to assist students develop their reasoning, teachers require an understanding of the nuances of reasoning (Stylianides, Stylianides, & Shilling-Traina, 2013). Stylianides, Stylianides, and Philippou (2007) asserted that "[i]f teachers' knowledge of proof is fragile … it is likely that teachers will teach proof poorly or will not teach proof at all" (p. 146). Consequently, "teachers will be more likely to incorporate reasoning into their mathematics lessons if they understand the opportunities for creating student reasoning" (Davidson et al., 2018, p. 3). Similarly, Clarke et al.'s (2012) teacher survey revealed that "many students appeared to have little experience in the opportunity to conjecture, justify and generalise, or certainly to articulate these processes verbally or in writing" (p. 30).

Llinares (2013) drew together previous research on noticing to define it as a teacher's ability to "identify relevant aspects of the teaching situation; use knowledge to interpret the events, and establish connections between specific aspects of teaching and learning situations and more general principles and ideas about teaching and learning" (p. 79). Noticing of reasoning involves "attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings" (Jacobs, et al., 2010, p. 172). However, Bragg, et al. (2016) reported on teachers' noticing of their students' difficulties in articulating their reasoning. Such difficulties may inhibit teachers' interpretation of student understanding and prevent the appropriate responses Jacobs, et al. (2010) suggested.

In planning for reasoning, Davidson et al. (2018) emphasised the importance of "identifying reasoning potential in tasks; anticipating student responses; [and] eliciting reasoning through effective prompting" (p. 3). They suggested that open-ended tasks had the potential to allow a specific focus on the development of students' reasoning. However, for these tasks to be effective a teacher's ability to use appropriate prompts to elicit further reasoning is also important (Martino & Maher, 1999). So when planning for reasoning teachers may find Kilpatrick et al.'s (2001) advice useful, that is, choosing tasks that provide opportunities to conjecture, generalise and justify and develop a culture supporting reasoning in their classes.

The inclusion of mathematical reasoning in curriculum documents highlights the importance attributed to mathematical reasoning in the study of mathematics (Brodie, 2010). In the Australian Curriculum: Mathematics (AC: M) (ACARA, 2017) mathematical reasoning is embedded as one of the four key proficiency strands.

> Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. Students are reasoning mathematically when they explain their thinking, when they deduce and justify strategies used and conclusions reached (ACARA, 2017).

This review has highlighted previous research around mathematical reasoning and its assessment. The next section outlines the data collection and analysis of this study.

Methodology

The participants in this study are 34 primary teachers and their students at four Victorian schools who were involved in trialling tasks and resources for the reSolve project: Assessing Mathematical Reasoning. The teachers' experiences provide insights into the challenges they faced in attempting to assess the mathematical reasoning of their students. These teachers took part in a professional learning program consisting of a presentation on the nature of reasoning and its assessment; teaching and observing other teachers at their school teaching a task focussing on reasoning provided by the research team with a post-lesson discussion with the researchers where they attempted to assess the students' reasoning using a rubric developed by the research team. A second lesson focussing on reasoning was trialled. For this lesson the teachers sourced or created a task they considered would provide opportunities for students' reasoning. Once again, their lessons were observed by the other teachers at their school and at least two of the research team and followed by another post-lesson discussion with the researchers. The rubric used in this post-lesson discussion was a modified version which had been revised in consultation with the teachers at the previous post-lesson discussion. The post-lesson discussions were audio-recorded and transcribed and copies of students' work samples and associated rubrics were collected.

Thematic coding (O'Leary, 2014) was employed to examine and record patterns within the data. The thematic coding process involved "searching for patterns and interconnections; [then] mapping and building themes" (O'Leary, 2014, p. 331). All 18 transcripts of post-lesson discussions were read as a whole to gain an appreciation of the content, keeping in mind the search for challenges to assessing mathematical reasoning identified in the research literature. The transcripts were read for a second time to identify additional re-occurring ideas that emerged. These ideas were grouped together into themes. The results are structured according to the themes identified in the literature and the other themes which emerged from the repeated reading of the transcripts and analysis of the Australian Curriculum: Mathematics Year Level Descriptions.

Lack of direction/support in curriculum documents

A close inspection of the statements in AC: M regarding mathematical reasoning reveal limited direction/support in AC: M for teachers attempting to implement and assess reasoning in their classrooms since the Achievement Standards for each level do not include reasoning. The Year Level Descriptions revealed a focus on specific mathematical content rather than reasoning. For example, Foundation (ACF): "reasoning includes explaining comparisons of quantities, creating patterns and explaining processes for indirect comparison of length" (ACARA, 2017). Level 1 – length and data; Level 2 – calculations and data; Level 3 and 4 – number, angles, and data; Level 5 – fractions, decimals, and data; Level 6 – mental calculations, number sequences, geometry, and data.

Some participants expressed the need for direction from the curriculum documents. For example:

> **School C Gloria**: I thought the lesson is actually really good, the discussion and stuff. So this [assessment], you know we're always looking at, you know [checking] against the AusVELS … how does this relate to the levels? … If I knew whereabouts all of this stuff was plotted in the continuum that would help me.

> **School B Clare:** Yeah but is it [the rubric] AUSVELS or is it just your own?

These quotes indicate teachers were expecting that the assessment of reasoning would have been included in the curriculum documents.

Teacher knowledge of reasoning

Teachers' degree of understanding of mathematical reasoning was evident in many transcripts. For example, the following quote suggests that Lisa is unsure about the nature of mathematical reasoning.

> **School C Lisa:** I think if you're doing it as you go around the class and you really [need to] know what each of these things mean. Because to make that quick decision, yeah that's where you are.

Many teachers focused on explaining as the most visible reasoning action. The following quote illustrates this teachers' confusion between understanding and reasoning.

> **School A Cathy**: I think Xxxx because she was explaining it to Yyyyy and Yyyyy is quite a lot lower and Xxxxx was using her explanations and when I pressed her on it she did have a grasp of it but sometimes she would use it. … It's been really developing because sometimes she will be in the class being like the expert too. So it's helping her explaining her reasoning and explain how she gets it and teach other kids. I had her in grade 3 and there is no way she would have been able to explain why – she knew she could do it but she could never explain it so it's a big step up.

Other teachers commented on the support they found in the rubric in extending their knowledge of the various aspects of mathematical reasoning.

> **School C Kerry:** Even just reading over it [rubric], I found it really clear, and therefore easily able to identify and put them [students] in, it's really clear on where they're at with it, it's not even beginning, they're developing … I found it [rubric], that I could actually, in my head I could classify my kids as we went.

> **School C Connie:** No, [there isn't a simpler structure that will help teachers] because I think the amount of information you've got in there helps you when you're making judgments with the kids.

The following quote includes instances where text from the rubric has been used to support their explanations of students' reasoning and so build their knowledge of reasoning.

> **School C Robyn**: I've only got through half of them but … most of them are at developing and consolidating. This task … had a little bit of everything in it. It had a bit of analysing because they had to recall and repeat a pattern, and had forming conjectures and generalising because they had to explain the meaning of the rule and justifying logical argument because they had to say why and because.

Teacher noticing of reasoning

Whilst many teachers noticed students' reasoning they experienced difficulties in understanding that reasoning. For example:

School B Clare: I'm not quite sure. [There's] an equation – I'm not quite sure what he's trying to say? He's saying small numbers make 10s and big numbers make hundreds. So if you had 57 plus 75 I don't even know where this comes from?

School A Rosie: The fact that she knows how many to add each time and she's realised the pattern of both of them, would you say that that's what that refers to or is that too advanced?

The following exchange shows teachers working together to try to make sense of a students' reasoning.

School B Terry: [it's difficult] to describe exactly what he did there because they've just said you know 'I've counted with my fingers and the number chart so I know the answer' and then he's provided place value but he's actually done it for ones like that and just represented the number but not his actual calculation of how he got that number. So described what he did. He recognised it was incorrect using materials, objects or words but it wasn't really coherent.
School B Clare: It's not that he's not coherent, he is coherent he's just not – he's only got one argument really and then he's done the place value. Well that's two arguments I suppose.

These teachers noticed that the student was reasoning, but were struggling to identify how he was reasoning or what reasoning actions were evident.

Students' difficulties in articulating their reasoning

Some teachers observed the challenges students faced in communicating their reasoning. In the following quote the teacher acknowledges this struggle and suggests the student tries to verbalise their reasoning before trying to write it down.

School C Ann: I gave them [a chance] … to explain this as well orally because I said sometimes you don't say exactly what you want to say when you write, so have a go at explaining.

Another teacher commented on a students' lack of mathematical language limiting their ability to clearly explain their reasoning.

School B Clare: Here's a low one, even he did really well. I got his because he was the one that used the words like – the numbers were too heavy. He said 'that's too heavy to have these lighter numbers'. He didn't use mathematical language he used his understanding of number-

These excerpts from the data provide evidence in this data set of the challenges already identified in the research literature. In addition, two further challenges were noted in the data for this study.

Inadequacy of work samples

When considering work samples teachers frequently referred to conversations they had had in class with the students. These remembered conversations emphasised the need to listen to students' attempts to articulate reasoning.

School C Kerry: I'd say I must've picked up quite a bit from just the conversations as well because they haven't actually shown it here [on the work sample]. So it seems like from the conversations on the floor they were watching and listening [to each other].

This view was wide spread amongst the participating teachers and re-iterated at other schools. For example:

School D Elizabeth: it's hard just looking at their random working out.

School A Rosie: I think her verbal explanation is very good but probably didn't have time to write it. I think with the next session she will. She'll just be given that extra time to work on the formula.

Difficulty in tracking and reporting student progress in reasoning

Many teachers commented on the complexity of mathematical reasoning and the difficulties they faced in trying to observe and record individual student's progress especially since this was not necessarily evident in the work samples. They concluded that it was too hard to consider the reasoning of all students in a single lesson so planned to concentrate on just a few using the short version of the rubric.

> **School A June:** [Assessing reasoning] is time consuming. You really need to sit down and have a think about what it's asking you, but then if I was to use it [the rubric] I would just at a glance go, well [for these students] next time I'm going to do that. But I probably wouldn't sit there and go each one and say, well he can't do this one, so I'm going to do that. It's almost for me a little bit overwhelming because there's so much on the page.

Many teachers were concerned about summative assessment and rating students' reasoning performance. For example:

> **School B Terry:** That's how I score.

> **School C Con:** How would you mark someone that has terrible reasoning for one task and then really good for another? Does that mean they just know one task better than another reasoning task?

These results provide examples of six different themes evident in the data. This paper acknowledges the findings of previous research related to the challenges of assessing mathematical reasoning and extends this work by considering the data collected in the post lesson discussions for this project.

Discussion

Whilst in the AC year level descriptions there is a focus on content areas, they are also to some degree consistent with education research literature related to mathematical reasoning. There is a strong emphasis on communicating reasoning in the year level descriptions of reasoning. Brodie (2010) reported that reasoning may be communicated to others verbally, written or through representations or an internal individual explanation to oneself in an attempt to clarify our own thinking (Foundation, Level 1, Level 2, Level 3 and Level 4). Pedemonte (2007) described mathematical reasoning as explaining comparisons, processes and strategies (Level 4) and is required to interpret and evaluate others' representations, conjectures, explanations (Foundation, Level 1, Level 6) and justify (Level 1, Level 6) strategies and results. There is less emphasis in the AC:M on generalising which considered to be fundamental to mathematics (Mata-Pereira, & da Ponte, 2017), with Carpenter et al. (2003) stressing the importance of providing students with opportunities to explore, generalise, form and test conjectures.

Like the research of Clarke et al. (2012), Herbert et al. (2015), Loong et al. (2017), and Hilton et al. (2016) challenges related to teachers' knowledge of various aspects of reasoning is also evident in this data for this study. However, Jacobs et al. (2010) reported the positive effects of a professional development project focused on algebraic reasoning. A consequence of gaps in teachers' knowledge of reasoning is the difficulties teachers face in noticing, interpreting and assessing students' reasoning. Consistent with previous research, teachers in this study did comment on problems students had explaining their reasoning (Bragg, et al., 2016).

The data from this study confirmed the challenges already reported in the literature, but also revealed three new challenges. Firstly, the inadequacy of the year level descriptions of reasoning in the AC, but more importantly the challenges teachers face in trying to report on the development of reasoning of the students in their classes. To do this they need to be able to track each students' progress in some way. There was some discussion about how the reSolve rubric could be used in practice. Teachers comment that work samples alone were insufficient information on which to judge their students' reasoning. They insisted that there was a need to listen to students attempts to articulate reasoning, so perhaps just focusing on a few students each lesson.

Conclusion

Mathematical reasoning is now acknowledged in the Australian Curriculum as one of the four key ideas arching across all content areas. However, assessment of reasoning can be difficult. This paper sought to bring together many challenges faced by primary teachers as they attempt to assess mathematical reasoning. Some of these have been identified in the literature whilst two additional challenges were recognised by the participants in this study. No evidence could be found in the data for this study regarding the importance of planning to promote reasoning including task selection (Davidson et al., 2018). However, this remains an area rich in opportunities for further research.

The challenges evident in the data for this study and previously identified in the research literature are:

- Teacher knowledge of reasoning (Clarke et al., 2012; Herbert et al., 2015; Loong et al., 2017);
- Teacher noticing of reasoning (Jacobs et al., 2010);
- Students' difficulties in articulating their reasoning (Bragg, et al., 2016);

Additional challenges to assessing students' mathematical reasoning revealed in the document analysis and the data from the teachers are:

- Lack of direction/support in curriculum documents;
- Inadequacy of work samples
- Difficulty in tracking and reporting student progress in reasoning;

Whilst these challenges have been identified in the literature and this study, further research is needed to shed light on strategies which may mitigate these challenges. It is important to make the assessment of reasoning easier to do and more effective in tracking change in the development of students' reasoning, such as using the resolve rubric to assess just a few students each lesson.

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